Mixedness and teleportation

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Abstract

We show that on exceeding a certain degree of mixedness (as quantified by the von Neumann entropy), entangled states become useless for teleporatation. By increasing the dimension of the entangled systems, this entropy threshold can be made arbitrarily close to maximal. This entropy is found to exceed the entropy threshold sufficient to ensure the failure of dense coding.

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Shared bipartite entanglement has found a host of interesting applications in quantum communications [1–3]. It is natural to expect that the efficiency of these applications would go down with the decrease of shared entanglement. However, apart from the degree of entanglement of a shared state, there is another physical factor, namely the mixedness of the state, which causes deterioration of the efficiency of the applications. Though for given classes of states (such as the Werner states [4]), the entanglement of the state may decrease with the mixedness of the state, the two are not necessarily related concepts. For example, a mixed state can have more entanglement than a completely pure (zero mixedness) disentangled state. Thus we are interested in how the mixedness of a given state, taken as an independent physical criterion, affects the efficiency of the entanglement applications. In particular, we will focus on teleportation [3].

A good measure of mixedness of a state ρ is its von Neumann entropy [5] $S(\rho) = -\mathrm{Tr}(\rho \log \rho)$. We will first show that when the entropy of a given $N \times N$ state exceeds $\log N + (1 - \frac{1}{N}) \log(N+1)$, the state becomes useless for teleportation. To this end we will first need to prove a short theorem. For this theorem we need a quantity called the singlet fraction introduced by the Horodeckis [6]. The singlet fraction $F(\rho)$ of a $N \times N$ state ρ is defined as $\max\langle\Psi|\rho|\Psi\rangle$, where the maximum is taken over all the $N \times N$ maximally entangled states. We now proceed to our theorem.

Theorem: *If the entropy $S(\rho)$ of a state ρ of a $N \times N$ system exceeds $\log N + (1 - \frac{1}{N}) \log(N+1)$, then the singlet fraction $F(\rho) < \frac{1}{N}$.*

Proof: Let, for a certain state ρ, $F(\rho) \geq \frac{1}{N}$. This means that there exists, at least one $N \times N$ maximally entangled state $|\Psi_{\mathrm{Max}}\rangle$, for which $\langle\Psi_{\mathrm{Max}}|\rho|\Psi_{\mathrm{Max}}\rangle \geq \frac{1}{N}$. Let us write the state ρ as

$$\rho = \sum_{i=1,j=1}^{N^2} c_{ij} |i\rangle\langle j|, \tag{1}$$

where $\{|i\rangle\}$ is a basis formed from $|\Psi_{\mathrm{Max}}\rangle$ and $N^2 - 1$ other maximally entangled states. From the definition of singlet fraction it follows that the largest of the elements c_{ii} (say this is c_{11}) has a value greater than or equal to $\frac{1}{N}$. Now, we know that the von Neumann entropy

$S(\rho)$ of the state ρ is always less than or equal to its Shannon entropy in any particular basis. This implies

$$S(\rho) \leq -\sum_{i=1}^{N^2} c_{ii} \log c_{ii}. \tag{2}$$

Subject to the constraint $c_{11} \geq \frac{1}{N}$, the expression $-\sum_{i=1}^{N^2} c_{ii} \log c_{ii}$ attains its highest value when $c_{11} = \frac{1}{N}$ and the rest $N^2 - 1$ elements c_{ii} are all equal. Thus

$$\begin{aligned} -\sum_{i=1}^{N^2} c_{ii} \log c_{ii} \leq & -\frac{1}{N} \log \frac{1}{N} \\ & -(1-\frac{1}{N}) \log\{\frac{1}{N^2-1}(1-\frac{1}{N})\} \\ & = \log N + (1-\frac{1}{N}) \log(N+1). \end{aligned} \tag{3}$$

From Eqs.(2) and (3) it follows that

$$S(\rho) \leq \log N + (1-\frac{1}{N}) \log(N+1). \tag{4}$$

Thus we have

$$F(\rho) \geq \frac{1}{N} \Longrightarrow S(\rho) \leq \log N + (1-\frac{1}{N}) \log(N+1). \tag{5}$$

The implication in the above equation is equivalent to

$$S(\rho) > \log N + (1-\frac{1}{N}) \log(N+1) \Longrightarrow F(\rho) < \frac{1}{N}. \tag{6}$$

In Ref. [6] the Horodeckis have shown that singlet fraction $F(\rho) < \frac{1}{N}$ implies that one cannot do teleportation with ρ with better than classical fidelity. Thus when the entropy of a state exceeds $\log N + (1-\frac{1}{N}) \log(N+1)$, then by virtue of the theorem proved above, the state becomes useless for teleportation. Here, the phrase "useless for teleportation" means "useless for teleportation with better than classical fidelity". Note that this value of entropy is a minimum threshold. At values of entropy arbtrarily close to this but less, a state ρ is not forbidden to allow better than classical teleportation. For example, consider the generalized Werner state [7] $W_N(\epsilon) = \epsilon|\Psi_N\rangle\langle\Psi_N| + (1-\epsilon)\rho_{\mathrm{M}}$ of $N \times N$ dimensions where ρ_{M} is the corresponding maximally mixed state. When ϵ is infinitisimally greater than $\frac{1}{N}$ (which automatically ensures that the singlet fraction is $> \frac{1}{N}$) the state will allow teleportation better than classical, but its entropy will only be slightly below $\log N + (1-\frac{1}{N})\log(N+1)$.

An interesting consequence of our result is the fact that as the dimension N of the systems is increased, the entropy threshold becomes closer and closer to the maximal possible entropy of the state. In fact as $N \to \infty$, we have $\log N + (1-\frac{1}{N})\log(N+1) \to 2\log N$. Thus for systems of very large dimensions, *even an entropy extremely close to the maximal entropy is not sufficient to ensure the failure of teleportation*.

It is now interesting to compare the entropy sufficient to ensure the failure of teleportation with the entropy sufficient to ensure the failure of another application, namely, dense coding [2]. Dense coding with mixed states have been studied before [8,9], but here our target is to identify a degree of mixedness above which dense coding is bound to fail. Here again, failure of dense coding will mean its capacity being less than or equal to the classical communication capacity of $\log N$ bits per qu-N-bit. An upper bound to the capacity for dense coding with mixed signal states W_i occurring with probabilities p_i is given by the Holevo bound [10] $H = S(\sum p_i W_i) - \sum p_i S(W_i)$. The first expression $S(\sum p_i W_i)$ can attain at most a value of $2\log N$. Thus when the entropy $S(W_i)$ of a signal state exceeds $\log N$ we have $H \leq \log N$. Therefore an entangled state ρ will fail to be useful for dense coding when $S(\rho) > \log N$. This is also a minimum threshold. For example, for the state $W_N(\epsilon)$, we have $H = 2\log N - S(W_N(\epsilon))$ for standard Bennett and Wiesner scheme of dense coding and this can exceed $\log N$ for $S(W_N(\epsilon))$ slightly less than $\log N$. This threshold of $\log N$ is evidently much smaller than the threshold $\log N + (1-\frac{1}{N})\log(N+1)$ sufficient to ensure the failure of teleportation.

In this paper we have shown that there is a degree of mixedness after which a state becomes useless for teleportation. We have quantified this mixedness with the von Neumann entropy, but we could as well use the linear entropy $S_L = 1 - \mathrm{Tr}\rho^2$. In that case the threshold for failure of teleportation will be $1 - \frac{2}{N(N+1)}$. The fact that on increasing the mixedness of a state, dense coding fails before teleportation indicates that teleportation is "more robust" to external noise. Of course, our entropic criterion is only a *sufficient condition* for the failure of teleportation. However, entropic criteria can never be necessary for the failure of any entanglement application because they fail even for pure disentangled states. It would be easier to calculate the entropy of a state than to calculate its singlet fraction as no maximization is involved in the former calculation. Hence mathematically, our entropic criterion ($S > \log N + (1 - \frac{1}{N}) \log(N+1)$) is more convinient than the corresponding singlet fraction condition ($F < \frac{1}{N}$). How about the realtion between mixedness and entanglement itself? We know that for a Bell diagonal state ρ with only two non-zero eigenvalues, the distillable entanglement [11] is equal to $1 - S(\rho)$ [12]. Such a state would not be distillable if $S(\rho) \geq 1$. Is there such an entropy threshold sufficient to ensure the failure of entanglement distillation for an arbitrary $N \times N$ state? We leave that as an interesting open question.

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